Assignment $\frac{5}{2}$.

This homework is due on last day of classes, December 11.

This assignment is worth as much as a normal homework assignment in terms of course grade, and is not included in denominator of your course grade. That is, it's a freebie.

There are total 10 problems in this assignment. Each problem is worth 10 points. 90 points is considered 100%. If you go over 90 points, you will get over 100% for this homework and it will count towards your course grade (but not over 115%).

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much.

1. Algebraic axioms of $\mathbb{R}$. Quick cheat-sheet

Reminder. (Subsection 2.1.1) On the set $\mathbb{R}$ of real numbers there two binary operations, denoted by $+$ and $\cdot$ and called addition and multiplication, respectively. These operations satisfy the following properties:

(A1) $a+b=b+a$ for all $a, b \in \mathbb{R}$,
(A2) $(a+b)+c=a+(b+c)$ for all $a, b, c \in \mathbb{R}$,
(A3) there exists $0 \in \mathbb{R}$ s.t. $0+a=a+0=a$ for all $a \in \mathbb{R}$,
(A4) for each $a \in \mathbb{R}$ there exists an element $-a$ s.t. $a+(-a)=(-a)+a=0$,
(M1) $ab=ba$ for all $a, b \in \mathbb{R}$,
(M2) $(ab)c=a(bc)$ for all $a, b, c \in \mathbb{R}$,
(M3) there exists $1 \in \mathbb{R}$ s.t. $1 \cdot a=a \cdot 1=a$ for all $a \in \mathbb{R}$,
(M4) for each $a \neq 0$ in $\mathbb{R}$ there exists an element $1/a$ s.t. $a \cdot (1/a)=(1/a) \cdot a=1$,
(D) $a(b+c)=ab+ac$ and $(b+c)a=ba+ca$ for all $a, b, c \in \mathbb{R}$.

We also assume $0 \neq 1$. The number $a+(-b)$ is denoted by $a-b$. The number $a \cdot b^{-1}$ is denoted by a/b or $\frac{a}{b}$.

2. Exercises

Every equality you write in this section should be accompanied by a reference to the exact property (A1)–(A4), (M1)–(M4), or (D) of real numbers or a previously proved claim you are using.

(1) (a) Prove that there is only one 0. (*Hint:* Suppose there are two, 0_1 and 0_2. Consider the number 0_1+0_2.)
 (b) Prove that each $a \in \mathbb{R}$ has a unique opposite number $-a$. (*Hint:* Suppose there are two, $(-a)_1$ and $(-a)_2$. Consider the expression $(-a)_1+a+(-a)_2$.)
 (c) Prove that for every $a \in \mathbb{R}$, $-(-a)=a$. (*Hint:* Look at the definition of $-a$.)
 (d) Prove that the equation $a+x=b$ has a unique solution. (*Hint:* Assume that some specific x is a solution. Add $-a=-a$ to the equality.)

(2) (a) Prove that there is only one 1.
 (b) Prove that each nonzero $a \in \mathbb{R}$ has a unique inverse number a^{-1}.
 (c) Prove that for every $a \in \mathbb{R}$, $(a^{-1})^{-1}=a$ whenever $a \neq 0$.
 (d) Prove that the equation $ax=b$ has a unique solution whenever $a \neq 0$.

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(3) (a) Prove that for every $a \in \mathbb{R}$, $a \cdot 0 = 0$. (*Hint:* Consider $a \cdot 0 + a \cdot 0$.)
 (b) Prove that if $ab = 0$ then $a = 0$ or $b = 0$. (*Hint:* If $b \neq 0$, multiply the equality by $b^{-1} = b^{-1}$.)
 (c) Suppose some set $\mathbb{A}$ with operations $+$ and $\cdot$ satisfies A1–A4, M1–M3, D but not necessarily M4. Then is it still true that if $ab = 0$ then $a = 0$ or $b = 0$?

(4) Prove the following for every $a, b \in \mathbb{R}$.
 (a) $a \cdot (-1) = -a$.
 (b) $(-1) \cdot (-1) = 1$.
 (c) $(-a) \cdot (-b) = a \cdot b$.
 (d) $(-a)^{-1} = -(a^{-1})$ if $a \neq 0$.
 (e) $-(a + b) = (-a) + (-b)$.
 (f) $(ab)^{-1} = (a^{-1}) \cdot (b^{-1})$ if $a, b \neq 0$.
 (g) $-(a/b) = (-a)/b$ if $b \neq 0$.

(5) Let $a, b, c, d \in \mathbb{R}$ and $b, d \neq 0$. Prove that
 (a) $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$.
 (b) $\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + c \cdot b}{b \cdot d}$.

3. Order axioms of $\mathbb{R}$. Quick cheat-sheet

Reminder. (Subsection 2.1.5) Let $\mathbb{A}$ be a set with two operations $+$ and $\cdot$ satisfying A1–A4, M1–M3 and D (for example, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$). The set $\mathbb{P} \subset \mathbb{A}$ is called the set of *positive elements* if

(i) If $a, b \in \mathbb{P}$, then $a + b \in \mathbb{P}$,
(ii) If $a, b \in \mathbb{P}$, then $ab \in \mathbb{P}$,
(iii) If $a \in \mathbb{A}$, then exactly one of the following holds: $a \in \mathbb{P}$, $a = 0$, $-a \in \mathbb{P}$.

Then we write $a < b$, $b > a$ if and only if $b - a \in \mathbb{P}$; $a \leq b$, $b \geq a$ if and only if $b - a \in \mathbb{P} \cup 0$.

4. More exercises

Every inequality you write in this section should be accompanied by a reference to the exact property (i)–(iii), or a previously proved claim that you are using.

(6) Let $a, b, c \in \mathbb{R}$. Prove that
 (a) If $a > b$ and $b > c$ then $a > c$. (*Hint:* $a - c = (a - b) + (b - c)$.)
 (b) If $a > b$ then $a + c > b + c$. (*Hint:* Use (i).)
 (c) If $a > b$ and $c > 0$ then $ca > cb$. (*Hint:* $ca - cb = c(a - b)$.)
 (d) If $a > b$ and $c < 0$ then $ca < cb$.

(7) (a) Prove that if $a \in \mathbb{R}$ and $a \neq 0$ then $a^2 > 0$. (*Hint:* Consider three cases according to (iii).)
 (b) Prove that $1 > 0$.
 (c) Prove that if $a, b > 0$ then $a/b > 0$.

(8) For $a, b, c, d \in \mathbb{R}$, prove that
 (a) if $a < b$, $c \leq d$, then $a + c < b + d$,
 (b) if $0 < a < b$, $0 < c \leq d$, then $0 < ac < bd$,
 (c) Let $0 < a < b$ and $c < d < 0$. Give an example where $ac < bd$, an example where $ac > bd$, and an example where $ac = bd$.

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5. Yet more exercises

(9) On the set $\mathbb{N}$, consider two operations: $\oplus$ and $\odot$ defined as follows: $a \oplus b = ab$ and $a \odot b = a^b$.

(a) Do properties A1, A2 hold for $\oplus$? That is, is it true that $a \oplus b = b \oplus a$, and that $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ for all $a, b, c \in \mathbb{N}$?
(*Hint:* For this and further items, the main way to figure out questions is to write out expressions with $\odot$ and $\oplus$ in terms of "usual" operations, using definition of $\odot$ and $\oplus$)

(b) Do properties M1, M2 hold for $\odot$?

(c) Is there unit element with respect to $\odot$? That is, is there an element $1_\odot \in \mathbb{N}$ such that $1_\odot \odot a = a \odot 1_\odot = a$ for all $a \in \mathbb{N}$?

(d) Is there a *right* unit element with respect to $\odot$? That is, is there an element $1_r \in \mathbb{N}$ such that $a \odot 1_r = a$ for all $a \in \mathbb{N}$?

(e) Is there a *left* unit element with respect to $\odot$? That is, is there an element $1_\ell \in \mathbb{N}$ such that $1_\ell \odot a = a$ for all $a \in \mathbb{N}$?

(f) Is $\odot$ distributive over $\oplus$ *on the left*? That is, is it true that $(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$?

(g) Is $\odot$ distributive over $\oplus$ *on the right*? That is, is it true that $c \odot (a \oplus b) = (c \odot a) \oplus (c \odot b)$?

(10) In each case below, determine if P is a set of positive elements (i.e. whether it satisfies (i), (ii) and (iii)).

(a) $\mathbb{A} = \mathbb{Z}$, $P = \mathbb{N}$,

(b) $\mathbb{A} = \mathbb{Z}$, $P = -\mathbb{N}$,

(c) $\mathbb{A} = \mathbb{Q}$, $P = \{r \in \mathbb{Q} : \ r > 1\}$,

(d) $\mathbb{A} = \mathbb{Q}$, $P = \{r \in \mathbb{Q} : \ r > -1\}$,

(e) $\mathbb{A} = \mathbb{C}$, $P = \{z = x + iy \in \mathbb{C} : \ x > 0\}$,

(f) Prove that no subset of $\mathbb{A} = \mathbb{C}$ can serve as a set positive elements. (In other words, one cannot endow $\mathbb{C}$ with a meaningful order.)